FACULTY
OF SCIENCES

25TH OF JANUARY 2025

# THE TRACE DENSITY BETWEEN DENSITY MATRICES

## A NIFTY TOOL IN NEW-PHYSICS SEARCHES

GHENT
UNIVERSITY

# SUMMARY

(credits to Joscha)


[arXiv]

- density matrix = all physical information
- sensitive to new physics
- trace distance most sensitive tool

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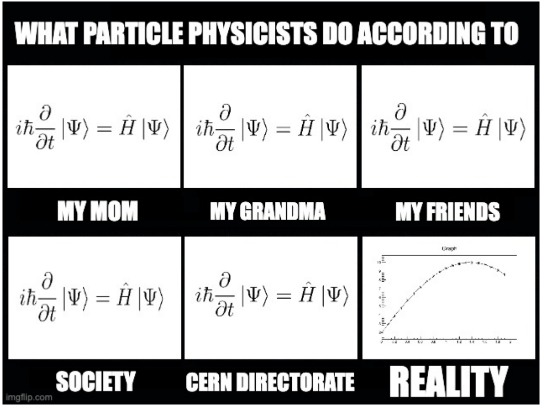
- density matrix $=$ all physical information
- sensitive to new physics
- trace distance most sensitive tool
- **but first: quantum mechanics 101**



# How to do quantum mechanics

(1) Observable $\hat{O}$ (e.g. spin)

(2) Describe system

(3) recipe expectation value

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(2) density $\rho = \sum_{ij} \rho_{ij} |\psi_i\rangle\langle\psi_j|$

(3) $\langle\hat{O}\rangle = \mathrm{Tr}\left(\rho\hat{O}\right) = \rho_{ij} O_{ji}$

$$|\Psi\rangle \equiv \begin{bmatrix} c_0 & c_1 & \dots \end{bmatrix}^T \qquad \langle\Psi| \equiv \begin{bmatrix} c_0^* & c_1^* & \dots \end{bmatrix}$$

$$\hat{O} \equiv \begin{bmatrix} O_{00} & O_{01} & \dots \\ O_{10} & O_{11} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \qquad \rho \equiv \begin{bmatrix} \rho_{00} & \rho_{01} & \dots \\ \rho_{10} & \rho_{11} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

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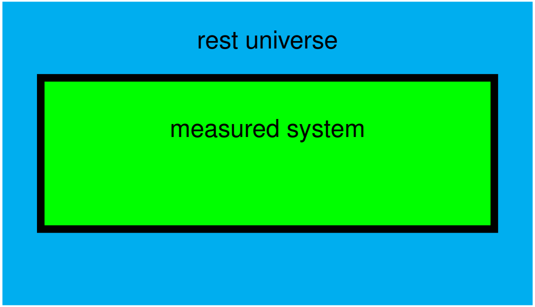
Why bother with the density?

# Mixed systems

wave function describes **closed** system

total universe



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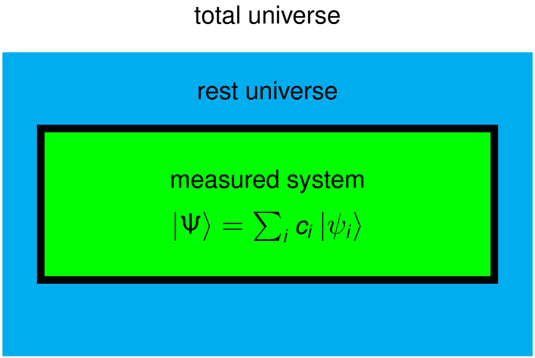

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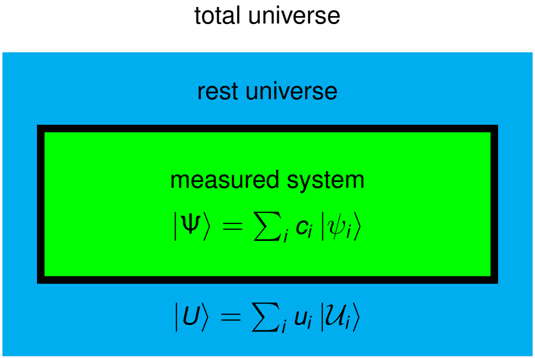

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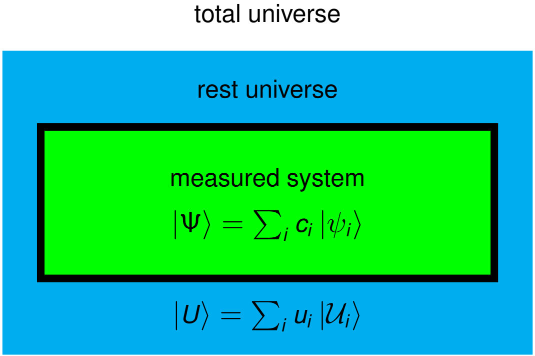


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observable $\hat{O}$ on system:

$$\langle \hat{O} \rangle = \langle \Phi | \hat{O} | \Phi \rangle = \langle \Psi | \hat{O} | \Psi \rangle \underbrace{\langle U | U \rangle}_{=1}$$

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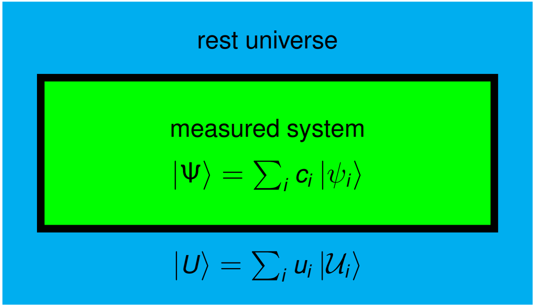


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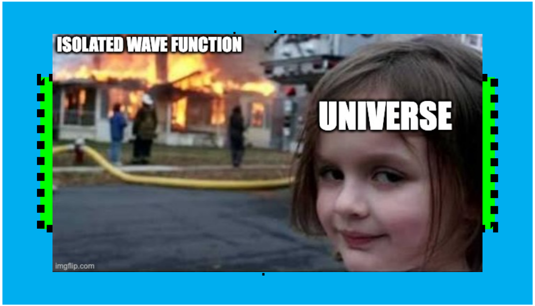


$$|\Phi\rangle = \sum_{ij} f_{ij} |\psi_i\rangle |\mathcal{U}_j\rangle$$

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$$= \sum_{ijk} f^*_{kj} f_{ij} \langle \psi_k | \hat{O} | \psi_i \rangle$$

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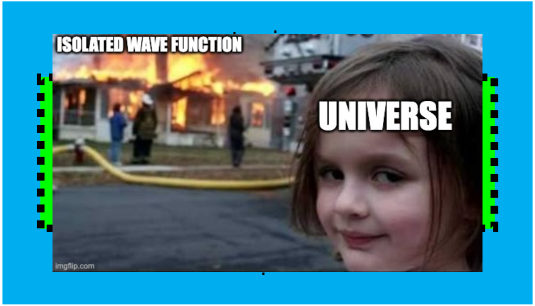


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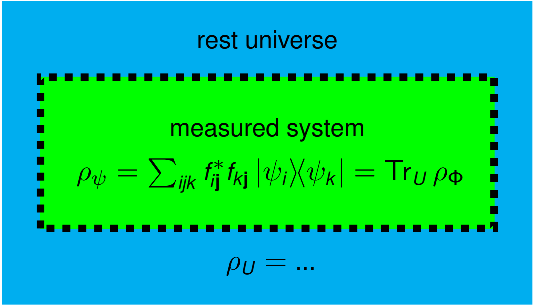



$$\rho_\Phi = \sum_{ijk\ell} f_{ij} f^*_{k\ell}\,|\psi_i\rangle\,|\mathcal{U}_j\rangle\,\langle\psi_k|\,\langle\mathcal{U}_\ell|$$

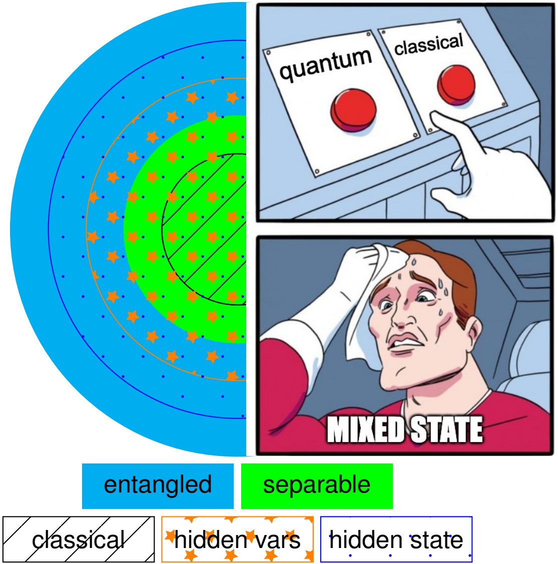
# MIXED SYSTEM: QUANTUM EFFECTS

Pure state: only two types

- classical product state $|\psi\rangle\,|\phi\rangle$
- entangled states: $\sum_i |\psi_i\rangle\,|\phi_i\rangle$

Mixed state: rich categorization / observables

$\rightarrow$ concurrence, discord, magic...



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*example: qubit*

spin observable

$\vec{\Sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$

$$\rho = \frac{1}{2}\left(\mathbb{1}_2 + \sum_i S_i\sigma_i\right)$$

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$\mathrm{Tr}(\mathbb{1}_2) = 2, \mathrm{Tr}\,\sigma_i = 0$

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- product states: $|\phi\rangle\,|\psi\rangle \rightarrow \rho_\phi \otimes \rho_\psi$

  $\rightarrow$ $(A\otimes B)_{ijkl} = A_{ij}B_{kl}$
  - $\operatorname{Tr}(A\otimes B) = \operatorname{Tr}A\operatorname{Tr}B$

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individual spins

$\mathbb{1} \equiv$ no measurement

$$\rho = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \sum_i B_i^0 \sigma_i \otimes \mathbb{1} + \sum_i B_i^1 \mathbb{1} \otimes \sigma_i + \sum_{ij} C_{ij} \sigma_i \otimes \sigma_j\right)$$

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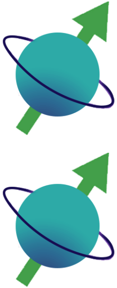
$\mathrm{Tr}(\mathbb{1}_2 \otimes \mathbb{1}_2) = \mathrm{Tr}\, \mathbb{1}_4 = 4$

measure both spins

$\sim$ correlations



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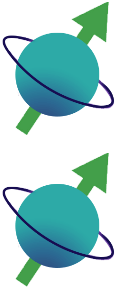

Correlation $\propto \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ so $C_{ij} \equiv$ correlation only if unpolarized ($B^{0,1} = 0$)

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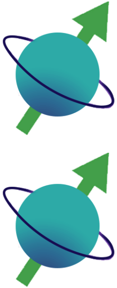

*alternatively, $\hat{\Sigma}_z$ eigenstates*

$$\rho = \sum_{ijk\ell} r_{ij}\,|s_i s_j\rangle\langle s_k s_\ell| = \begin{pmatrix} s_{00\,00} & s_{00\,01} & s_{00\,10} & s_{00\,11} \\ s_{01\,00} & s_{01\,01} & s_{01\,10} & s_{01\,11} \\ s_{10\,00} & s_{10\,01} & s_{10\,10} & s_{10\,11} \\ s_{11\,00} & s_{11\,01} & s_{11\,10} & s_{11\,11} \end{pmatrix}$$

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$= \langle\mu^+\mu^-|\,\hat{\mathcal{T}}\,|e^+e^-\rangle\,\langle e^+e^-|\,\hat{\mathcal{T}}\,|\mu^+\mu^-\rangle$

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- diagonal elements $\equiv$ differential xsec
- $\text{Tr}\,\rho_{ee}^{\text{out}} \equiv \frac{d\sigma}{d\Omega}(e^+e^- \to X)$ *(unnormalized)*

# Cross sections: $e^+e^-$ with spin

1. unpolarized, uncorrelated initial state: $\rho_{ee}^{\text{in}} = \frac{1}{4}\sum_{s_e^+ s_e^-} |s_e^+ s_e^-\rangle\langle s_e^+ s_e^-|$

$$\rho_{ee}^{\text{in}} = \frac{1}{4}\left( \mathbb{1}\otimes\mathbb{1} \underbrace{\begin{matrix} +\sum_i B_i^{\text{in}+}\sigma_i\otimes\mathbb{1} \\ +\sum_i B_i^{\text{in}-}\,\mathbb{1}\otimes\sigma_i \\ +\sum_{ij} C_{ij}^{\text{in}}\,\sigma_i\otimes\sigma_j \end{matrix}}_{=0} \right) = \frac{1}{4}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

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2. transition $\rho_{ee}^{\text{in}} \to \rho_{ee}^{\text{out}} = \hat{\mathcal{T}}^\dagger \rho_{ee} \hat{\mathcal{T}}$ and take final state $\mu^+\mu^-$

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(!) normalized density: (quantum) probability density (on the diagonal)

# COMPARING DENSITIES

***Trace distance***

$$\mathcal{D}^T(\rho, \zeta) = \frac{1}{2}\,\mathrm{Tr}\,\sqrt{(\rho - \zeta)^\dagger(\rho - \zeta)}$$

***Fidelity***

$$\mathcal{F}(\rho, \zeta) = \mathrm{Tr}\,\sqrt{\sqrt{\rho}\,\zeta\sqrt{\rho}}$$

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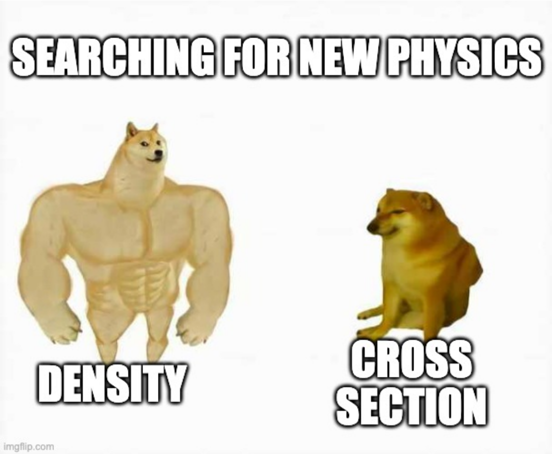


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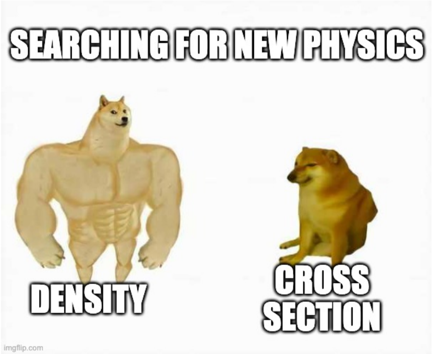
$\phi\rangle|$

new physics searches using $D^T(\rho, \rho_{\mathrm{SM}})$ or $\mathcal{F}(\rho, \rho_{\mathrm{SM}})$

# Searching new physics

$$\chi^2(\lambda) = \left(\frac{\mathcal{D}^T[\rho_{\rm NP}(\lambda) - \rho_{\rm SM}]}{\sigma_{\mathcal{D}^T}}\right)^2 + \left(\frac{\sigma_{\rm NP} - \sigma_{\rm SM}}{\sigma_\sigma}\right)^2$$



# Example: Top quark chromomagnetic dipole moment

$$\mathcal{L}_{\text{dip}} \propto g_s \mu_t \times \underbrace{\bar{t}\sigma^{\mu\nu} q_\nu \overbrace{G_\mu}^{\text{gluon field}} t}_{\text{tensor vertex=dipole}}$$

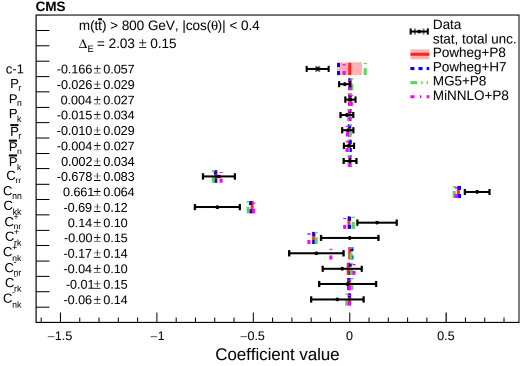
or equivalently in SMEFT
(before EW symmetry breaking)

$$\mathcal{L}_{\text{dip}} = \frac{c_{tG}}{\lambda^2}(\mathcal{O}_{tG} + \mathcal{O}_{tG}^\dagger)$$

Benchmark limit:

- inclusive xsec
- LHC (7 and 8 TeV) and tevatron (1.96 TeV)
- $-0.046 \leq \mu_t \leq 0.040$

CMS run 2 measurement



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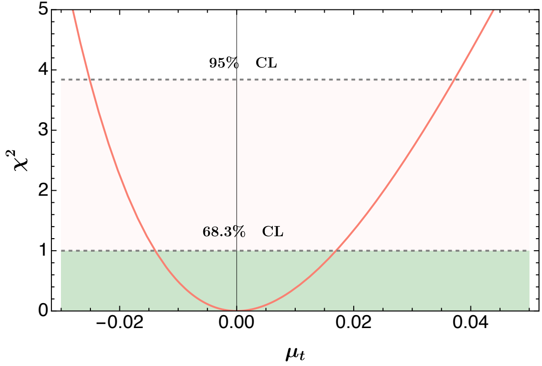
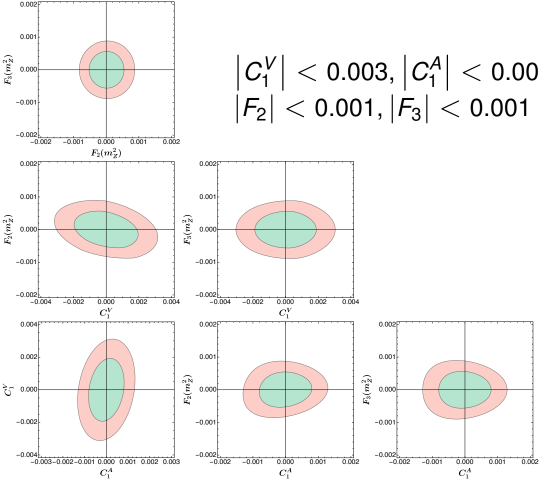



$$-0.025 \leq |\mu_t| \leq 0.037$$

# EXAMPLE: ANOMALOUS $\tau$ COUPLING AT FCC-$ee$

$$\chi^2(\mathrm{NP}) = \left(\frac{\mathcal{D}^T[\rho_{\mathrm{NP}} - \rho_{\mathrm{SM}}]}{\sigma_{\mathcal{D}^T}}\right)^2 + \left(\frac{\sigma_{\mathrm{NP}} - \sigma_{\mathrm{SM}}}{\sigma_{\mathcal{D}^T}}\right)^2$$

$$\mathcal{L}^{\mathrm{NP}}_{\tau Z} \propto \overline{\tau} \begin{bmatrix} \gamma^\mu q^2/m_\tau^2 \left(C_1^V + \gamma_5 C_1^A\right) \\ +i\sigma^{\mu\nu} q_\nu/(2m_\tau)\left(F_2 + \gamma_5 F_3\right) \end{bmatrix} \tau Z_\mu$$

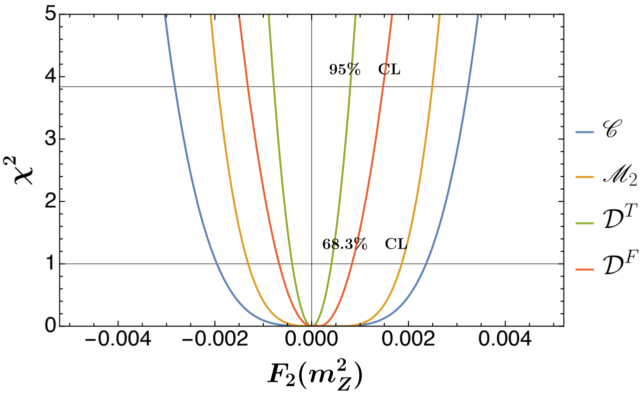



$\left|C_1^V\right| < 0.003, \left|C_1^A\right| < 0.001$
$\left|F_2\right| < 0.001, \left|F_3\right| < 0.001$

Benchmark limit:

- FCC-ee simulation at $s = m_Z^2$
- combines concurrence, incl xsec, ...
- $\left|C_1^V\right| < 0.01, \left|C_1^A\right| < 0.001$
  $\left|F_2\right| < 0.003, \left|F_3\right| < 0.001$

# Comparison of different metrics


* $\mathcal{C}$ *measures entanglement*

* $\mathcal{M}_2$ *measures diffculty to classically simulate*


THANK YOU FOR LISTENING!

# APPENDIX

GHENT UNIVERSITY

# QUANTUM PROBABILITY

positive definite matrix: $\rho \geq 0$

- $\rho_{ij} = \sum_k c_{ik} c_{jk}^* = (\mathcal{C}\mathcal{C}^\dagger)_{ij} \rightarrow \langle\phi| \rho |\phi\rangle = \left\langle C^\dagger\phi \middle| C^\dagger\phi \right\rangle \geq 0$
- diagonalizable: $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ with $p_i \geq 0$
- system in state $\phi_i$ with probability $p_i$

$\rightarrow$ quantum probability distribution

classical merchanics: density matrix *always* diagonal