Chaotic Bohmian trajectories for stationary states

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Introduction

Bohmian mechanics is an alternative approach to quantum mechanics which is deterministic and non-local, i.e. particles follow definite and predictable trajectories that depend on the configuration of the whole universe. In this work, we investigate the emergence of chaos in Bohmian trajectories for **stationary wave functions** [1].

Questions

Does $\psi(\mathbf{r},\cancel{t})$ allow for chaos in 3d or in 2d many-particle systems ?

Entanglement and complexity of the wave function $\overset{?}{\leftrightarrow}$ chaos

Bohmian mechanics

Actual point-particles of position $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n)$ moving under the influence of the wave function [2]

- Wave function ψ satisfies the usual Shrödinger equation
- Particles trajectories are determined by the guiding equation

$$\frac{d\mathbf{r}}{dt} = v^\psi(\mathbf{r},t)$$

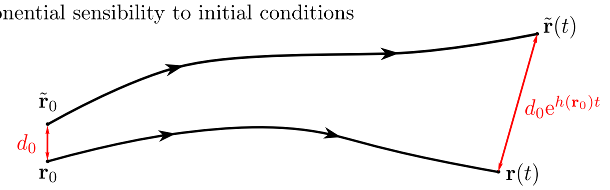
with velocity field $v^\psi = (\mathbf{v}_1^\psi, \ldots, \mathbf{v}_n^\psi)$ given by ($\hbar = 1$)

$$\mathbf{v}_k^\psi(\mathbf{r},t) = \frac{1}{m_k}\Im\left(\frac{\nabla_k\psi(\mathbf{r},t)}{\psi(\mathbf{r},t)}\right)$$

Can lead to the same predictions as standard quantum mechanics

Characterizing chaotic trajectories

Exponential sensibility to initial conditions



Quantifying chaos : **Lyapunov exponent** [3]

$$h(\mathbf{r}_0) = \lim_{t\to\infty}\left(\lim_{|\mathbf{d}(t_0)|\to 0}\frac{1}{t}\ln\left(\frac{|\mathbf{d}(t)|}{d_0}\right)\right)$$

Poincaré-Bendixon theorem [3]

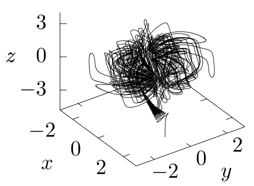
Chaos only possible if $n - c \geq 3$ with n the number of degrees of freedom and c the number of constants of motion

Constants of motion for stationary wave functions

We consider

$$\psi(\mathbf{r},t) = e^{iEt}\phi(\mathbf{r}) = e^{iEt}\sum_i c_i\phi_i(\mathbf{r}) \Rightarrow v^\psi(\mathbf{r},\cancel{t})$$

$c_i \in \mathbb{C}$ and $\phi_i(\mathbf{r})$ are degenerate eigenstates of the system Hamiltonian and constitute an orthogonal basis of the Hilbert space



Eigenstates in spherical coordinates

$$\phi_i(\mathbf{r}) = f_i(r)g_i(\theta)e^{im_i\varphi}$$

with f_i and g_i real functions

Example : Common eigenstates of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ for a 3d harmonic oscillator

Results : Stationary states allow for chaos

Symmetries of the wave function ⇒ constant of motion

Example : The state

$$\phi(\mathbf{r}) = \phi_a(\mathbf{r}) + \phi_b(\mathbf{r}) = f_a(r)g_a(\theta)\chi_a(\varphi) + f_b(r)g_b(\theta)\chi_b(\varphi)$$

admits the constant of motion $C = F(r) - G(\theta)$ with $dF(r)/dr = f(r)$, $dG(\theta)/d\theta = g(\theta)$ and

$$f(r) = \frac{1}{r^2}\frac{f_a f_b}{f_a\partial_r f_b - f_b\partial_r f_a}, \qquad g(\theta) = \frac{g_a g_b}{g_a\partial_\theta g_b - g_b\partial_\theta g_a}$$

Energy ≠ constant of motion

The energy of the system is the same for any initial conditions and thus does not constrain the Bohmian trajectories.

Chaos and complexity of the wave function

Stationary wave functions of the 3d harmonic oscillator ($n = 3$)

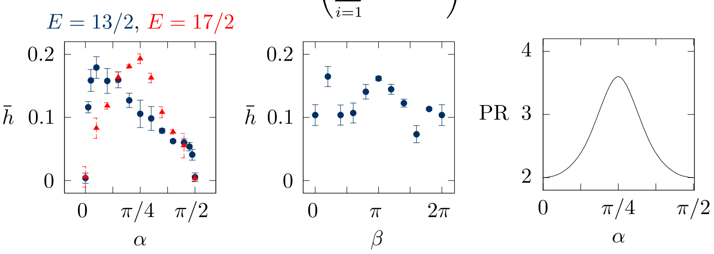
$$\begin{aligned}\phi(\mathbf{r};\alpha,\beta) &= \sum_{i=1}^{4}|c_i(\alpha)|e^{i\phi_i(\beta)}\phi_i(\mathbf{r})\\ &= \mathcal{N}(\alpha)\Big(\cos\alpha\,\phi_1(\mathbf{r}) + e^{i(\beta+\pi/3)}\sin\alpha\,\phi_2(\mathbf{r})\\ &\quad + e^{i(2\pi\cos\beta+\pi/5)}\cos^2\alpha\,\phi_3(\mathbf{r}) + e^{i(-2\beta+\pi/7)}\sin^2\alpha\,\phi_4(\mathbf{r})\Big)\end{aligned}$$

$\alpha = 0$ or $\pi/2 \Rightarrow$ constant of motion $\Rightarrow$ no chaos

Quantifying the complexity of the wave function

Measure of the degree of superposition using the **participation ratio** [4]

$$\mathrm{PR}(\psi) = \left(\sum_{i=1}^{N}|\langle\phi_i|\psi\rangle|^4\right)^{-1}$$



Average of the Lyapunov exponent $\bar{h}$ as a function of α and β

Chaos and entanglement

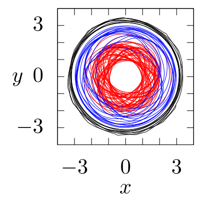
W entanglement class (3 particles, $n = 6$)

$$\begin{aligned}\phi_{\mathrm{W}}^{3\mathrm{p}}(\alpha,\beta,\gamma) =& \sqrt{\alpha}\phi_{a,1}\phi_{a,2}\phi_{b,3} + \sqrt{\beta}\phi_{a,1}\phi_{b,2}\phi_{a,3}\\ &+ \sqrt{\gamma}\phi_{b,1}\phi_{a,2}\phi_{a,3} + \sqrt{\delta}\phi_{a,1}\phi_{a,2}\phi_{a,3}\end{aligned}$$

$\phi_{i,k}(r_k,\varphi_k)$ ($i = a, b$ and $k = 1, 2, 3$) degenerate eigenstates of the 2d harmonic oscillator, $\alpha, \beta, \gamma \geq 1$ and $\delta = 1 - (\alpha + \beta + \gamma) \geq 0$

Example : $\phi_{\mathrm{W}}^{3\mathrm{p}}(1/3, 1/3, 1/3)$

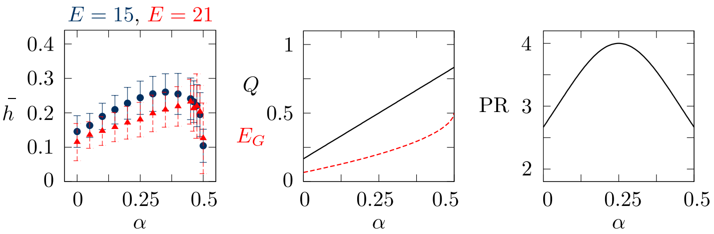
1 constant of motion (r_1, r_2, r_3) and $h \approx 0.11$



Entanglement measures

- **Geometric entanglement** E_G [5]
  Distance to the set of separable states
  $E_G(\phi_{\mathrm{W}}^{3\mathrm{p}}(\alpha, 1/2 - \alpha, 1/2)) = 1/2$ for all values of α
- **Meyer-Wallach** Q [6]
  Average of the linear entropies of each qubit

Compararison with average of the Lyapunov exponent $\bar{h}$:



Conclusion

Stationary states ⇒ chaotic Bohmian trajectories

provided the effective number of degrees of freedom $n - c$ is greater than 3

Complexity of the wave function, entanglement and chaos

- Participation ratio is a good indicator of the amount of chaos
- Meyer-Wallach measure of entanglement displays the best qualitative agreement with the amount of chaos

However, PR and Q do not depend on the intrinsic complexity of basis states which can modify the amount of chaos

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