Fixed Order QCD Calculations

or

The Magic of Perturbation Theory at Colliders

Ben Page

University of Ghent

BND Graduate School

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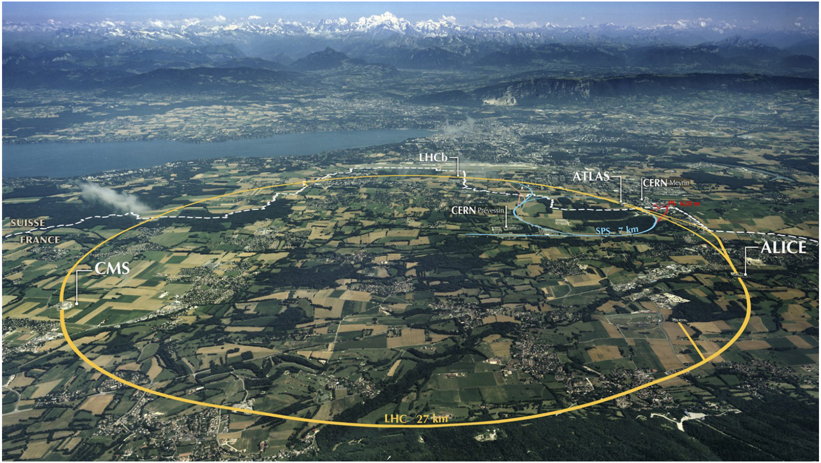
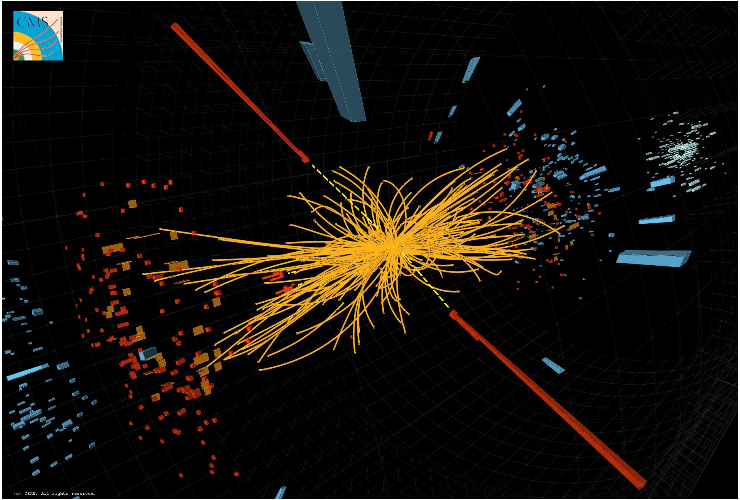
Lecture Content

1. Introduction to Predictions at Colliders
   - Fundamentals: Fields/Interactions, Partons/Asymptotic Freedom
   - Cross-section breakdown: Collinear Factorization + PDFs, Perturbative Expansion, Real-Virtual Decomposition.
2. Modern Techniques for Feynman Diagrams
   - Tree-level: Helicity/Color, Recursive Methods
   - One-loop: Regularization, Master Integral Reduction, Unitarity
   - Two-loop (and beyond): Frontiers and difficulties.

Introduction

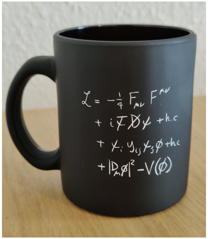

LHCb
ATLAS
CERN Meyrin
CERN Prévessin
SPS 7 km
SUISSE
FRANCE
CMS
ALICE
LHC 27 km

A Collision at the LHC

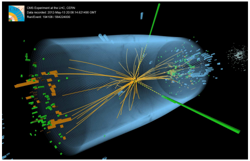


Precise Perturbative Predictions

- To understand collider data, we need precise SM predictions.



$\longrightarrow$



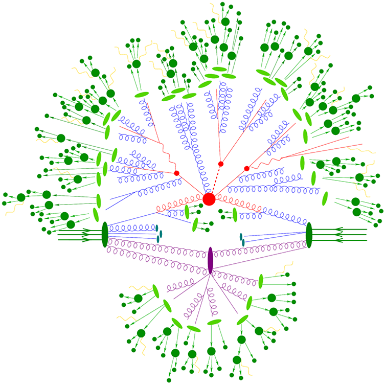
- Perturbation theory is our major tool for making predictions.

$$\sigma[\alpha_S] \sim \sigma_{\mathrm{LO}} + \alpha_S \delta\sigma_{\mathrm{NLO}} + \alpha_S^2 \delta\sigma_{\mathrm{NNLO}} + \mathcal{O}(\alpha_S^3).$$

- As a LHC is a proton machine, we will focus on QCD.

A Physical Picture for *pp* Scattering

- Proton-proton collision. Initial state: non-perturbative QCD.
- High energy (fundamental) interactions occur in the centre.
- Produced particles emit QCD radiation and hadronize.



How can we build a quantitative description of the collision, making use of perturbation theory?

QCD Fundamentals

Quantum Chromodynamics: A Field Theory

- 8 gluon fields $A^1_\mu, \dots A^8_\mu$, $3N_f$ quark fields $\psi^f_1, \dots, \psi^f_3$.
- Interactions neatly described by Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}} = \underbrace{\overline{\psi^f}_i (i\partial\!\!\!/ - m_f)\psi^f_i}_{\mathcal{L}_{\mathrm{Dirac}}} - \underbrace{\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a}_{\mathcal{L}_{\mathrm{YM}}} - \underbrace{g_s \overline{\psi^f_i} A\!\!\!/^a T^a_{ij} \psi^f_j}_{\mathcal{L}_{q/g-\mathrm{int}}}.$$

where $$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu,$$

f_{abc}, T^a_{ij} = $SU(N_c)$ group theory factors.

QCD: "The $SU(N_c)$ gauge theory of quark/gluon interactions".
$N_c = 3$ "colours", $N_f = 6$ flavours of quark.

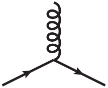
Building Blocks of Perturbative QCD

- Field propagators

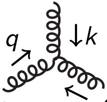
i  j $= i\delta_{ij}\dfrac{1}{\not{p} - m}$

a, μ  b, ν $= -i\delta_{ab}\dfrac{g^{\mu\nu}}{p^2}$

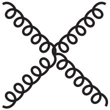
- Interaction vertices:



$= ig\gamma^{\mu} T^{a}_{ij},$



$$= gf^{abc}[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}]$$



$$= -ig^2[f^{abe}f^{cde}g^{\mu\rho}(g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})],$$

Asymptotic States in QCD

- Asymptotic states in perturbation theory are from free theory:

ψ_i  $\longleftrightarrow \; u(p)$ $\qquad A^a_\mu$  $\longleftrightarrow \; \varepsilon_\mu(p)$

- For QCD the free fields are the quarks and gluons.

- But! Asymptotic states of the interacting theory are hadrons.

First question when using perturbative QCD: fundamental degrees of freedom do not appear as asymptotic states!

The QCD Coupling and Asymptotic Freedom

- Coupling constants in QFT run! Captured by "β-function":

$$\beta = \mu^2 \frac{\partial}{\partial \mu^2} \alpha_S(\mu^2) = \frac{\alpha_S^2}{2\pi}\left(b_0 + \left[\frac{\alpha_S}{2\pi}\right] b_1 + \cdots\right), \qquad \alpha_S = \frac{g_S^2}{4\pi}.$$

- At leading order (one-loop), introducing reference scale μ_0:

$$\frac{1}{\alpha_S(\mu)} = \frac{1}{\alpha_S(\mu_0)} + \frac{b_0}{2\pi} \log(\mu^2/\mu_0^2)$$

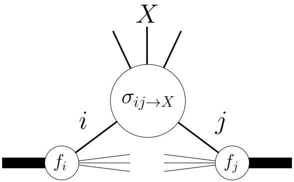
- In QCD $b_0 > 0$, so $\lim_{\mu\to\infty} \alpha_S(\mu) = 0$.

$$b_0 = \frac{11}{3} N_c - \frac{2}{3} N_f.$$

- $\alpha_S(M_z) \sim 0.12 \Rightarrow$ pQCD applicable at hadron colliders!

Collinear Factorization

- At high energy, proton-proton scattering "factorizes" into scattering of underlying partons (quarks + gluons).



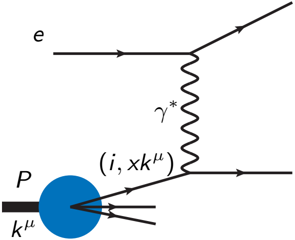
$$\longrightarrow \ \mathrm{d}\sigma = \sum_{i,j} \int \mathrm{d}x_i \mathrm{d}x_j f_i(x_i) f_j(x_j) \mathrm{d}\sigma_{i,j}(x_i, x_j).$$

- $\mathrm{d}\sigma_{pp}$ built from $\mathrm{d}\sigma_{i,j}$ and "parton distribution functions".

- This is an approximation. Valid up to $O(\Lambda_{\mathrm{QCD}}/Q)$.

- Must include parton shower/hadronization. [Marius' lectures]

Parton Distribution Functions



$f_i(x, Q^2)$: "probability of finding parton i with momentum fraction x when probing proton (P) at scale Q^2".

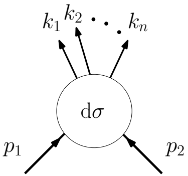
- Not perturbatively calculable! Determined from data.
- PDFs satisfy (perturbatively calculable) evolution equations.

$$Q^2 \frac{\partial}{\partial Q^2} f_i(x, Q^2) = \sum_j \frac{\alpha_S(Q^2)}{2\pi} \int_x^1 \mathrm{d}z \underbrace{P_{ij}(z)}_{\text{splitting functions}} f_j(x/z, Q^2).$$

[Dokshitzer, Gribov, Lipatov, Altarelli, Parisi]

Building Blocks of Perturbative Predictions

Partonic Cross Sections



$$\sigma = \frac{1}{2s} \int \mathrm{d}\Phi_n |\mathcal{A}(p_1, p_2, k_1, \dots k_n)|^2$$

$$\mathrm{d}\Phi_n = \prod_i \frac{\mathrm{d}^3 k_i}{(2\pi)^3 E_i} (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_i k_i \right)$$

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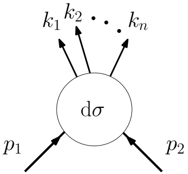

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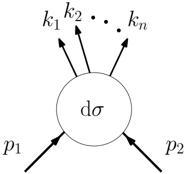

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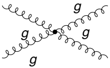
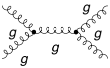
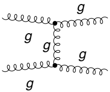
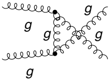
$\mathcal{A}_{gggg} =$  $+$  $+$  $+$  $+ \underbrace{\mathcal{O}(g_s^4)}_{\text{loops}}$

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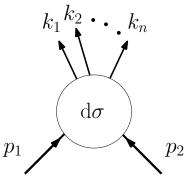

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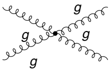
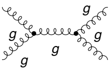
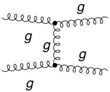
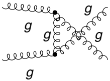
$\mathcal{A}_{gggg} =$  $+$  $+$  $+$  $+ \underbrace{\mathcal{O}(g_s^4)}_{\text{loops}}$

- NB: Squaring amplitude mixes contributions at higher orders

$$|\mathcal{A}^{(0)} + g_s^2 \mathcal{A}^{(1)} + \cdots|^2 = |\mathcal{A}^{(0)}|^2 + g_s^2\left[\mathcal{A}^{*(0)}\mathcal{A}^{(1)} + \mathcal{A}^{*(1)}\mathcal{A}^{(0)}\right] + \mathcal{O}(g_s^8).$$

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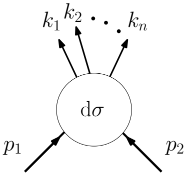

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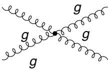
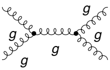
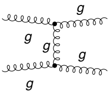
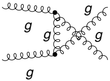
$\mathcal{A}_{gggg} =$  $+$  $+$  $+$  $+ \underbrace{\mathcal{O}(g_s^4)}_{\text{loops}}$

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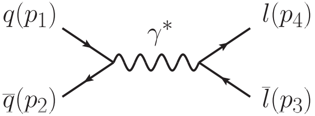
Amplitudes at Leading Order

- Let us consider the Drell-Yan process at leading order.

$$i\mathcal{A}(1^{q}_{s_1}, 2^{\overline{q}}_{s_2}, 3^{l}_{s_3}, 4^{\overline{l}}_{s_4}) =$$



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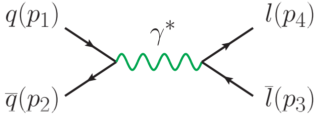

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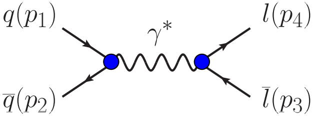

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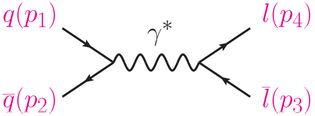

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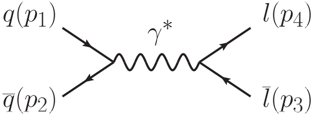

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- Average over states/color for unpolarized cross-section:

$$\frac{1}{4}\frac{1}{9}\sum_{s_k,i,j}|\mathcal{A}^2| = \frac{e_q^2 e_l^2 g^4}{12 s_{12}^2}\mathrm{tr}\left[(\not{p}_2 - m_q)\gamma^\mu(\not{p}_1 + m_q)\gamma^\nu\right]\mathrm{tr}\left[(\not{p}_4 + m_l)\gamma^\mu(\not{p}_3 - m_q)\gamma^\nu\right].$$

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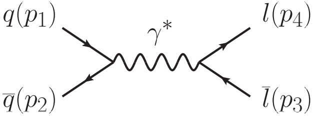

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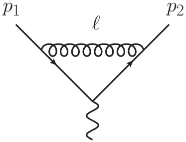
$$\frac{1}{4}\frac{1}{9}\sum_{s_k,i,j}|\mathcal{A}^2| = \frac{24e_q^2 e_l^2 g^4}{s_{12}^2}\left[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) + m_q^2(p_3\cdot p_4) + \cdots\right].$$

- Amplitude (and square) just rational functions of kinematics.

Amplitudes Beyond Leading Order

- Loop-corrections to scattering amplitude are integrals!



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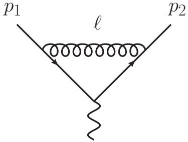

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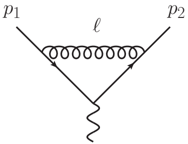

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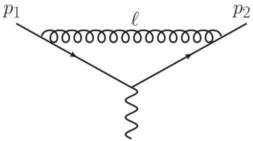
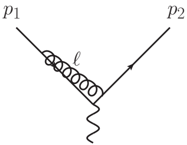
- Massless particles: unbounded contribution from infra-red.

$\ell \sim 0 \;\Rightarrow$



soft

$\ell \sim \lambda p_i \;\Rightarrow$



collinear

Cross-Sections at Next-to-Leading Order

Kinoshita-Lee-Nauenberg Theorem (KLN)

Must include contributions from physically indistinguishable σ.

$$\delta\sigma_{\text{NLO}} = \underbrace{2\int \mathrm{d}\Phi_n \mathrm{Re}\left[\mathcal{A}_n^{*\text{tree}}\mathcal{A}_n^{\text{1-loop}}\right]}_{\text{virtual}} + \underbrace{\int \mathrm{d}\Phi_{n+1}|\mathcal{A}_{n+1}^{\text{tree}}|^2}_{\text{real}}$$

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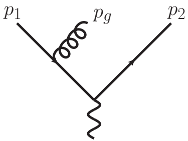
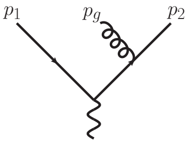
- Real contribution from singular emission. E.g. $e^+e^- \to q\bar{q}+g$:

$\mathcal{A}_{e^+e^-\to q\bar{q}g} =$  $+$ 

- Indistinguishable to $e^+e^- \to q\bar{q}$ if g is soft or collinear to $q/\bar{q}$.

Practical Infra-Red Divergences

- Divergences cancel between loop and phase-space integrals.

$$\underbrace{\delta\sigma_{\mathsf{NLO}}}_{\text{finite}} = \underbrace{\int \mathrm{d}\Phi_{n+1}|\mathcal{A}_{n+1}^{(0)}|^2}_{\text{divergent}} + 2\int \mathrm{d}\Phi_n \underbrace{\mathrm{Re}\left[\mathcal{A}_n^{*(0)}\mathcal{A}_n^{(1)}\right]}_{\text{divergent}}.$$

- Difficulty: how to perform Monte-Carlo numerically?

Practical Infra-Red Divergences

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- Difficulty: how to perform Monte-Carlo numerically?

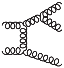
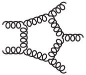
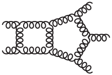
- Reorganize calculation: cancel divergences before integration.

$$\delta\sigma_{\text{NLO}} = \underbrace{\int d\Phi_{n+1}\left[|\mathcal{A}_{n+1}^{(0)}|^2 - S\right]}_{\text{finite}} + \int d\Phi_n \underbrace{\left[2\text{Re}\left(\mathcal{A}_n^{*(0)}\mathcal{A}_n^{(1)}\right) + \int d\Phi_1 S\right]}_{\text{finite}}.$$

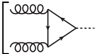
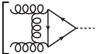
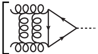
- Industry of subtraction/slicing methods. [Marius' Lectures]

Which Amplitudes for the LHC?

- $\alpha_S(\mu)$ grows as μ falls $\Rightarrow$ multi-jet processes prevelant.
- $\alpha_S(M_z) \sim 0.1$. Rule of thumb:
  - LO gives qualitative picture.
  - NLO gives quantitative picture.
  - NNLO reasonable error bars.
- Many processes have tree-level at leading order

$A_{5g} = \Big[$  $+ \cdots \Big] + \alpha_S \Big[$  $+ \cdots \Big] + \alpha_S^2 \Big[$  $+ \cdots \Big] + \mathcal{O}(\alpha_s^3).$

- NB: Loop induced processes have loop at leading order!

$A_{gg\to H} = \Big[$  $+ \cdots \Big] + \alpha_S \Big[$  $+ \cdots \Big] + \alpha_S^2 \Big[$  $+ \cdots \Big] + \mathcal{O}(\alpha_s^2).$

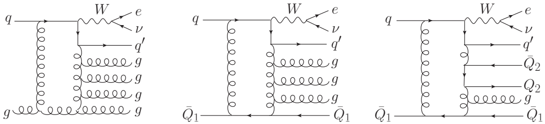
Feats of Perturbation Theory

- All multiplicity n-gluon amplitude known analytically:

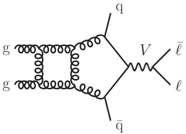
$$\mathcal{A}(1_g^-, 2_g^-, 3_g^+, \ldots, n_g^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle \langle n1 \rangle},$$

where $\langle ab \rangle = \overline{u}_-(k_a) u_+(k_b)$. [Parke, Taylor]

- NLO predictions for $W + 5j$.



- Two-loop amplitudes for $W + 2j$ production.



Part 1 Summary

- Factorization connects protonic to partonic scattering.

$$\mathrm{d}\sigma = \sum_{i,j} \int \mathrm{d}x_i \mathrm{d}x_j f_i(x_i) f_j(x_j) \mathrm{d}\sigma_{i,j}(x_i, x_j) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right).$$

- Beyond LO, infra-red divergences cancel between real/virtual.

$$\delta\sigma_{\mathrm{NLO}} = \underbrace{\delta\sigma^{(0)}_{n+1}}_{\text{real}} + \underbrace{\delta\sigma^{(1)}_{n}}_{\text{virtual}},$$

- Two major ingredients required for fixed order predictions:
  - Scattering amplitudes: Covered in this lecture.
  - Real/virtual cancellations: See Marius' lectures.

Fixed Order QCD Calculations

Part 2: Adventures in Perturbation Theory

Ben Page

University of Ghent

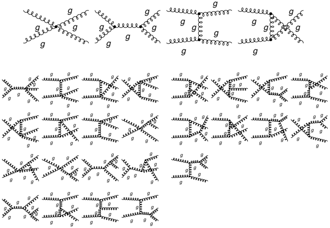
BND Graduate School

2^{nd} – 12^{th} September 2024

Tree-Level

Complexity of Feynman Diagram Approach

- Consider multi-parton scattering at tree level.
- Huge number of diagrams for high multiplicity.
- Diagram expressions large.



Process	$n = 7$	$n = 8$
$g\ g \to n\ g$	559,405	10,525,900
$q\bar{q} \to n\ g$	231,280	4,016,775

Major tools: “Quantum number management”, recursion relations.

Colour in Scattering

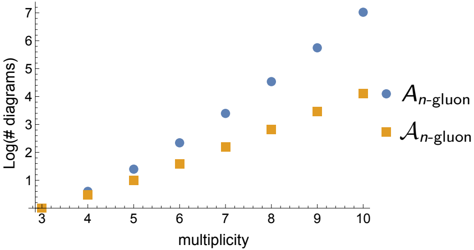
- Useful to break down amplitude into colour and kinematics.

$$A = \sum_i C_i \mathcal{A}_i .$$

- Many all multiplicity colour statements understood, e.g.

$$A^{(0)}_{n\text{-gluon}} = \sum_{\sigma \in S_n/\mathcal{Z}_n} \mathrm{tr}\left(T^{a_{\sigma_1}} T^{a_{\sigma_2}} \cdots T^{a_{\sigma_n}}\right) \mathcal{A}^{(0)}_{n\text{-gluon}}\left(\sigma_1, \sigma_2, \cdots \sigma_n\right).$$

- Individual $\mathcal{A}_i$ are easier to compute as fewer diagrams.

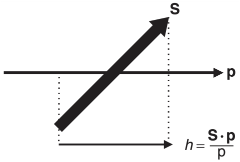


Helicity Amplitudes

- To calculate a scattering amplitude, must specify the state.

$$\epsilon^{\mu}_{s}(p) \qquad u_s(p) \qquad v_s(p) \qquad\qquad s = 1, 2.$$

- Distinguished set of states with well defined helicity:



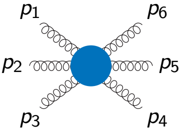
$$\epsilon^{\mu}_{s}(p) \to \epsilon^{\mu}_{\pm}(p)$$
$$u_s(p) \to u_{\pm}(p)$$
$$v_s(p) \to u_{\pm}(p).$$

- Amplitudes with helicity states have compact form!

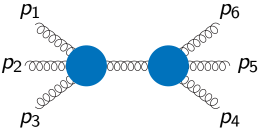
$$\mathcal{A}(1_g^-, 2_g^-, 3_g^+, \ldots, n_g^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle \langle n1 \rangle}.$$

On-Shell Recursion

- In "on-shell" limits amplitudes factorize:



$$= \frac{1}{s_{123}} \sum_{s}$$



$$+ \mathcal{O}(s_{123}^0)$$

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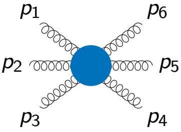

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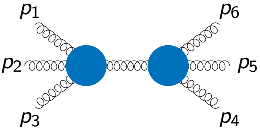

$$+ \mathcal{O}(s_{123}^0)$$

- Can be used to construct a recursion relation for amplitudes:

$$\mathcal{A}(1, \ldots, n) = \sum_k \sum_h \frac{\mathcal{A}_L(\tilde{1}, \ldots, -\tilde{P}_k^{\bar{h}}) \mathcal{A}_R(\tilde{P}_k^h, \ldots, n)}{\tilde{P}_k^2}.$$

[Britto, Cachazo, Feng, Witten]

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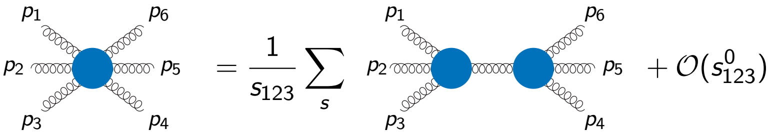

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[Britto, Cachazo, Feng, Witten]

- Very useful for building compact analytic results, e.g.

$$\mathcal{A}_{1_g^-,2_g^-,3_g^-,4_g^+,5_g^+,6_g^+} = \frac{1}{\langle 5|3+4|2]}\left(\frac{\langle 1|2+3|4]^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{234}} + \frac{\langle 3|4+5|6]^3}{[61][12]\langle 34\rangle\langle 45\rangle s_{345}}\right)$$

Off-Shell Recursion (i): Organizing Feynman Diagrams

Consider the trivalent diagrams contributing to $\mathcal{A}_5^{(\mathrm{tree})}$.

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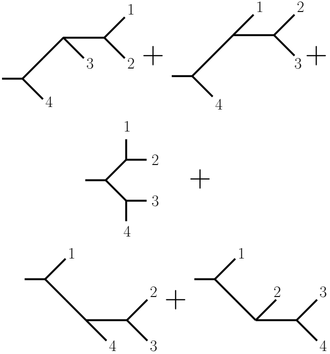

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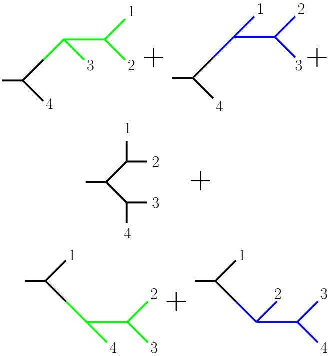

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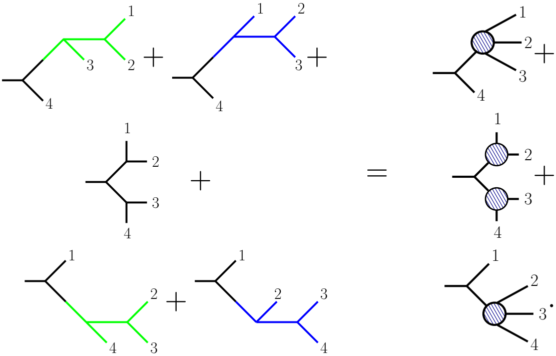

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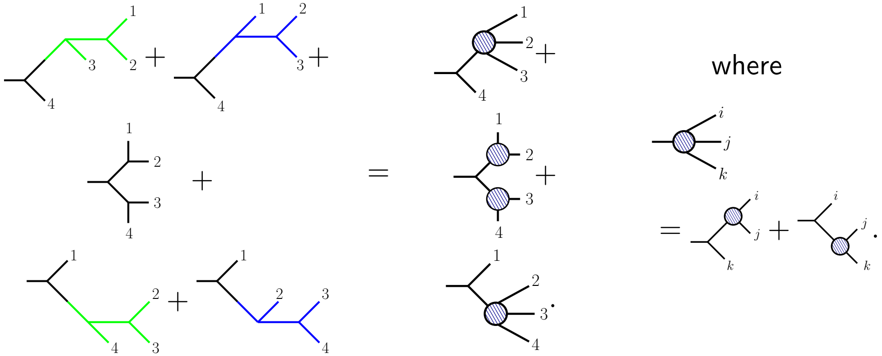

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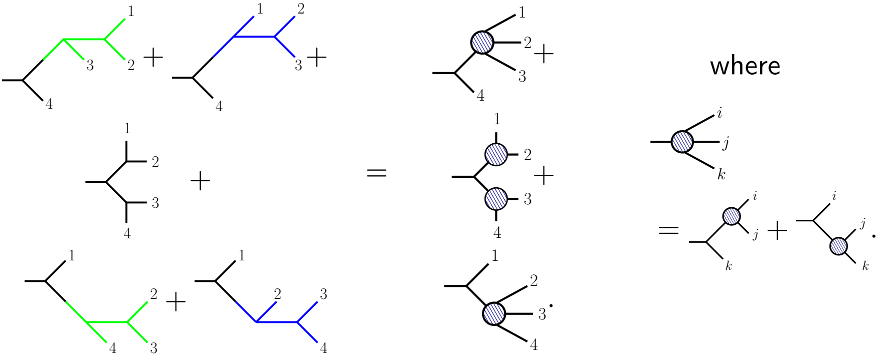

The diagram sum can be (recursively) organized into currents.

Off-Shell Recursion (ii)

Let us consider gluons, introducing the off-shell current $\mathcal{J}^\mu$

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$$\mathcal{A}(1, \ldots, n) = \lim_{p_n^2 \to 0} \epsilon_{\mu_n} p_n^2 \mathcal{J}^{\mu_n}(1, \ldots, n-1),$$

Off-Shell Recursion (ii)

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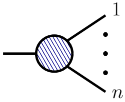
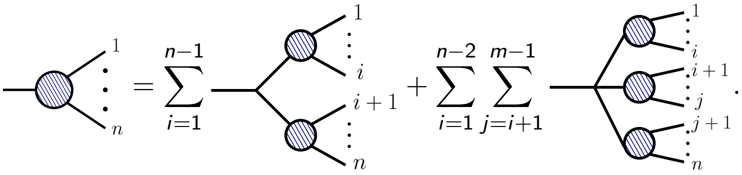
$$\mathcal{A}(1, \ldots, n) = \lim_{p_n^2 \to 0} \epsilon_{\mu_n} p_n^2 \mathcal{J}^{\mu_n}(1, \ldots, n-1),$$

$\mathcal{J}^\mu$ satisfies the "Berends-Giele" recursion relation.

$\mathcal{J}^\mu(1, \ldots, n) =$  .



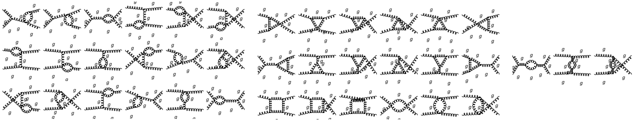
Efficient numerical implementation for high multiplicity $\mathcal{O}(n^4)$.

One-Loop

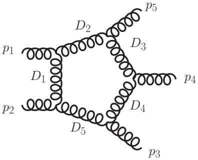
Loop Level Complexities

- Even larger diagrammatic combinatorics:

$$\mathcal{A}^{(1)}_{4g} \sim$$



- Moreover, each and every term is a Feynman integral.



$$= \int \mathrm{d}^4\ell \frac{N(\ell)}{D_1 D_2 D_3 D_4 D_5}.$$

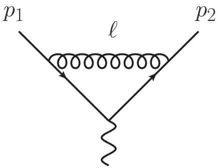
- How do we compute the integrals?
- How do we manage these large expressions?
- Can we build automated tools?

Dimensional Regularization

- Use "dim-reg" to tackle intermediate divergences.

$$\mathrm{d}^4\ell_i \quad \to \quad \mathrm{d}^D\ell_i, \qquad \text{where} \quad D = 4 - 2\epsilon.$$

- Divergences arise as poles in ϵ. E.g.



$$\sim \underbrace{\frac{1}{\epsilon^2}}_{\text{soft/collienar}} + \underbrace{\frac{\log(p_1 \cdot p_2)}{\epsilon}}_{\text{collinear}} + \mathcal{O}(\epsilon^0).$$

- Major blocker to use of Monte-Carlo integration.

- Take limit $D \to 4$ at end of calculation.

$$\sigma(D) = \sigma(4) + \mathcal{O}(\epsilon).$$

Master Integral Decomposition

We write $A^{(\text{loop})}$ in terms of a small set of master integrals:

$$A^{(\text{loop})}(p_1, \ldots, p_n) = \sum_k \underbrace{\mathcal{C}_k(p_1, \ldots, p_n)}_{\text{rational functions}} \underbrace{\mathcal{I}_k(p_i \cdot p_j, p_i^2, m_i^2)}_{\text{master integrals}}.$$

Building blocks:

- Coefficients $\mathcal{C}_k$: process dependent.
- Integrals $\mathcal{I}_k$: process independent, depend only on kinematics.

Divide and conquer approach

- How do we efficiently compute the rational functions?
- How do we numerically evaluate the master integrals?

Integral Reduction

Many integrals. However, controlled by Lorentz invariance!

$$\int \mathrm{d}^D \ell \frac{\ell^\mu \ell^\nu}{\ell^2(\ell - p)^2} = A g^{\mu\nu} + B p^\mu p^\nu .$$

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$$\int \mathrm{d}^D\ell \frac{\ell^\mu \ell^\nu}{\ell^2(\ell-p)^2} = A g^{\mu\nu} + B p^\mu p^\nu.$$

Find A, B by contracting the equation with $g^{\mu\nu}$ and $p^\mu p^\nu$:

$$\begin{pmatrix} \int \mathrm{d}^D\ell \frac{\ell^2}{\ell^2(\ell-p)^2} \\ \int \mathrm{d}^D\ell \frac{(\ell\cdot p)^2}{\ell^2(\ell-p)^2} \end{pmatrix} = \begin{pmatrix} D & p^2 \\ p^2 & p^4 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

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$$\begin{pmatrix} 0 \\ \frac{p^4}{4}\int \mathrm{d}^D\ell \frac{1}{\ell^2(\ell-p)^2} \end{pmatrix} = \begin{pmatrix} D & p^2 \\ p^2 & p^4 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

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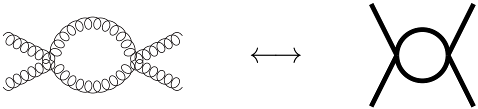
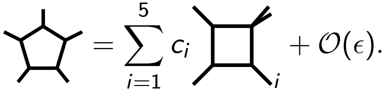
Gauge theory integrals same as scalar theory integrals!

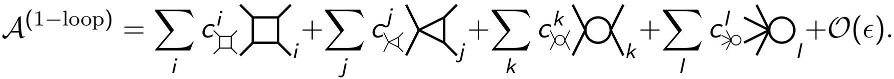


Master Integral Decomposition at One Loop

- External momenta are 4D ⇒ high-point integrals reduce, e.g.:



- Altogether, we see that we can write a one-loop amplitude as:



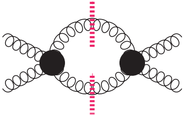
- Universal decomposition: valid for any process.

Efficient implementation of this decomposition was the missing ingredient that allowed for the "NLO revolution" around 2010!

Organizing by Unitarity

- Large number of terms. Break problem down by unitarity.

$$\mathrm{Disc}_{s_{12}}\left[\mathcal{A}^{(1)}_{gggg}\right] = \int \mathrm{d}\Phi \mathcal{A}^{(0)}_{gg\to gg}\mathcal{A}^{(0)}_{gg\to gg} =$$



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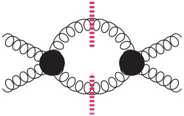
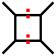
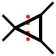
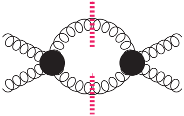


- Compare to discontinuity of MI decomposition:

$$\mathrm{Disc}_{s_{12}}\left[\mathcal{A}^{(1)}_{gggg}\right] = c_{1234}$$  $$+c_{(12)34}$$  $$+c_{12(34)}$$  $$+c_{(12)(34)}$$ 

Organizing by Unitarity

- Large number of terms. Break problem down by unitarity.

$$\mathrm{Disc}_{s_{12}}\left[\mathcal{A}^{(1)}_{gggg}\right] = \int \mathrm{d}\Phi \mathcal{A}^{(0)}_{gg\to gg}\mathcal{A}^{(0)}_{gg\to gg} =$$ 

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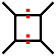
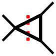
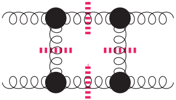
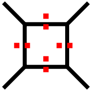
$$\mathrm{Disc}_{s_{12}}\left[\mathcal{A}^{(1)}_{gggg}\right] = c_{1234}$$  $+c_{(12)34}$  $+c_{12(34)}$  $+c_{(12)(34)}$ 

- (Integrand) factorization allows more fine-grained split up:

 $= c_{1,2,3,4}$ 

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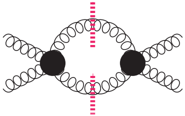
$$\mathrm{Disc}_{s_{12}}\left[\mathcal{A}^{(1)}_{gggg}\right] = \int \mathrm{d}\Phi \mathcal{A}^{(0)}_{gg\to gg}\mathcal{A}^{(0)}_{gg\to gg} =$$ 

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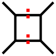
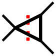
$$\mathrm{Disc}_{s_{12}}\left[\mathcal{A}^{(1)}_{gggg}\right] = c_{1234}$$  $+c_{(12)34}$  $+c_{12(34)}$  $+c_{(12)(34)}$ 

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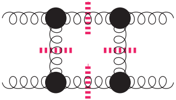
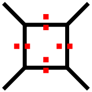
 $= c_{1,2,3,4}$ 

- Exploit tree-level advances. Reduction performed numerically!

Automation!

- Automated unitarity-based strategy for integral reduction:
  - NINJA, Samurai, CutTools.

- Many automatic tools for one-loop amplitude calculation*:
  - GoSAM
  - HELAC-1Loop
  - OpenLoops
  - MadGraph5_AMC@NLO
  - Blackhat
  - Njet
  - Recola
  - NLOX

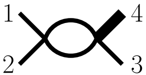
  . . .

* Many more developments: off-shell recursion for integrands, expansions around singular configurations for stability, etc...

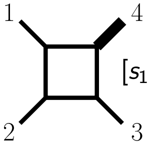
Numerical computations of high multiplicity (up to $\sim 2 \to 6$) one-loop amplitudes are readily available!

One Loop Master Integrals

- One loop integrals well understood. At worst dilogarithms:



$$= \frac{1}{\epsilon} + 2 + \log(-s) + \mathcal{O}(\epsilon).$$



$$[s_{12}s_{23}] = \frac{2}{\epsilon^2}\left[(-s_{12})^{-\epsilon} + (-s_{12})^{-\epsilon} - (-p_4^2)^{-\epsilon}\right] - \log^2\left(\frac{s_{12}}{s_{23}}\right) - \frac{\pi^2}{3}$$
$$- 2\mathrm{Li}_2\left(1 - \frac{p_4^2}{s_{12}}\right) - 2\mathrm{Li}_2\left(1 - \frac{p_4^2}{s_{23}}\right) + \mathcal{O}(\epsilon).$$

$$s_{ij} = (p_i + p_j)^2$$

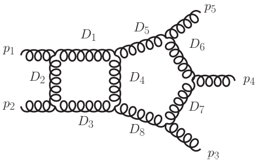
- Essentially all integrals necessary for collider physics known.
- Scalar integrals compiled in many libraries: `LoopTools`, `Golem95C`, `OneLOop`, `QCDLoop 2.0`, `Collier`.

See e.g. [https://arxiv.org/pdf/1912.06823] for summary.

Two-Loop (and Beyond)

New Difficulties at Two Loops

- Two-loop diagrams now $\sim$ 8-fold integral!



$$= \int \mathrm{d}^4\ell_1 \mathrm{d}^4\ell_2 \frac{N(\ell_1, \ell_2)}{D_1 \cdots D_8}.$$

- Soft/collinear divergences are more severe. (Up to $\frac{1}{\epsilon^4}$).
- Lorentz invariance insufficient to reduce to masters.
- Master integrals unknown. Computation mathematically deep.

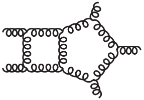
Calculations handled case by case, understanding is built as we go.
Throw every trick we have at it: reduction, unitarity, approximate...

Leading Colour Approximation

- In the leading-colour approximation, amplitudes can simplify.

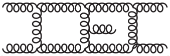
- Consider 5-gluon amplitude

$\mathcal{A}^{(2)}_{5g} =$



$\underbrace{}_{\mathcal{O}({N_c}^2)}$

$+$



$\underbrace{}_{\mathcal{O}({N_c}^1)}$

$+\, \mathcal{O}(10000)$ diagrams.

- Often, complicated non-planar integrals are sub-leading in N_c.

- $\mathcal{O}(10\%)$ effect on $\delta\sigma_{\text{NNLO}}$. Important to study!

$$\delta\sigma_{\text{NNLO}} = \delta\sigma^{\text{LC}}_{\text{NNLO}} + \mathcal{O}\left(\frac{1}{N_c^2}\right).$$

Master Integrals and Differential Equations

Aim: numerically evaluate ϵ expansion: $\mathcal{I}_k(\vec{p},\epsilon) = \sum_{n=-4}^{\infty} \mathcal{I}_k^{(n)} \epsilon^n$

- Master integrals satisfy differential equations.

$$\mathrm{d}\mathcal{I}_k = \mathbf{M}_{kl}(\epsilon, s_{ij})\mathcal{I}_l.$$

- If integrals generalize logarithms can expose ϵ structure

[Gehrmann, Remiddi '01; Henn '13]

$$\mathrm{d}\mathcal{I}_k = \underbrace{\epsilon}_{\text{regulator}} \underbrace{\tilde{\mathbf{M}}_{kl}(s_{ij})}_{\text{differential forms}} \mathcal{I}_l.$$

- ϵ-factorization facilitates expansion around 4d limit:

$$\mathcal{I}_k^{(n)} = \sum_{\alpha,l} \int \tilde{\mathbf{M}}_{kl} \mathcal{I}_l^{(n-1)} + \text{constant}.$$

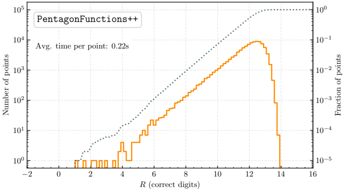
Numerically Solving the Differential Equation

Pentagon functions: [Gehrmann, Henn, lo Presti '18]

- Dedicated iterated integral code:

$$\mathcal{I}_k^{(n)} \sim \int_0^1 \mathrm{d}\log(W_n[t_n]) \cdots \int_0^{t_2} \mathrm{d}\log(W_1[t_1]).$$

- Very efficient for "high" multiplicity.



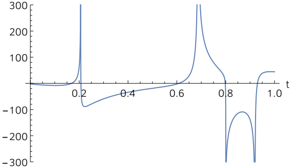
[Chicherin, Sotnikov '20]

Series expansions: popularized by [Moriello '19]

- Patch together h_k from power series:

$$\mathcal{I}_k^{(n)} \sim \sum_{j_1,j_2} (t-t_0)^{j_1/2} \log(t-t_0)^{j_2}.$$

- Codes: DiffExp/SeaSyde/AMFlow.
  [Hidding '20] [Armadillo et al '22] [Liu, Ma '22]



[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

Integration By Parts

- Lorentz-invariance not enough to reduce to master integrals.
- Further relations from "fundamental theorem of calculus":

$$\int_a^b \mathrm{d}x F'(x) = F(b) - F(a).$$

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$$\int_R d\omega = \int_{\partial R} \omega.$$

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$$\int_R d\omega = 0.$$

Must take into account "integration-by-parts" relations:
For Feynman Integrals total derivatives integrate to zero.

Reducing Two-Loop Amplitudes to Masters

- Reduction strategy for two-loops more pedestrian:
  - Total derivatives induces many relations between integrals.
  - Solve the linear system via Gauss elimination.
- Many public programs for IBP reduction: `FIRE`, `FiniteFlow`, `NeatIBP`, `LiteRed`, `Reduze`, `KIRA`...
- NB: Analytic algorithm $\Rightarrow$ all two-loop results are analytic.

Large bottleneck for high multiplicity: solving relations analytically.

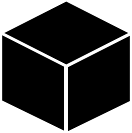
Rethinking Computer Algebra: Analytic Reconstruction

- Analytic coefficients built from numerical samples via Ansatz.

$$\mathcal{C}_k(p_1, \ldots, p_n) = \sum_{j=1}^{N_k} c_{jk} \mathfrak{a}_{jk}(p_1, \ldots, p_n), \qquad c_{jk} \in \mathbb{Q}.$$

- Numerical evaluations provide constraints on unknown c_{jk}.

$$(p_1^{(0)}, \ldots, p_n^{(0)}) \longrightarrow$$



$$\longrightarrow \mathcal{C}_1(p_1^{(0)}, \ldots, p_n^{(0)}).$$

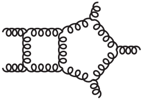
- Made practical by finite field methods (working modulo p).
  [Schabinger, von Manteuffel '14; Peraro '16]

- Sidesteps complex algebra – only intermediate numerics!

State-of-the-Art Two-Loop Amplitudes

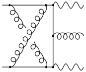
Five-Point *NNLO* QCD (no internal masses)

$pp \to 3j$



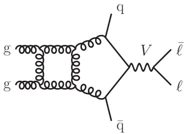
[Abreu et al], [Agarwal et al], [De Laurentis et al]

$pp \to \gamma\gamma + j$



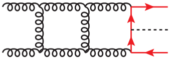
[Agarwal et al], [Chawdhry et al], [Badger et al]

$pp \to W + 2j$



[Abreu et al] [Badger et al] ($b\bar{b}$)

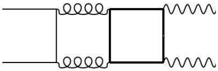
$H + b\bar{b}$



[Badger et al]

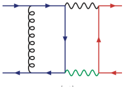
Four-Point QCD/EW @ NNLO (internal masses)

$pp \to \gamma\gamma$



[Bonciani et al]

$pp \to \gamma\gamma + j$

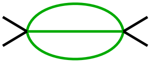


[Armadillo et al]

Outlook

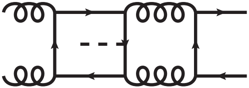
Elliptic Integrals

- At two-loops, functions go beyond logarithm generalizations.
- Example: two-loop sunrise integral, contains "elliptic" curve.



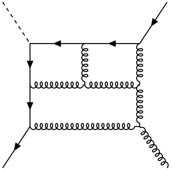
$$\sim \int_0^1 \mathrm{d}x \frac{\log(\chi(x,y))}{y}, \qquad \underbrace{y^2 = \prod_{i=1}^{4}(x-a_i)}_{\text{elliptic curve}}.$$

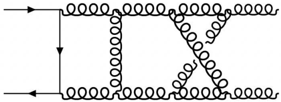
- No general understanding of special functions.
- Arise frequently in cases with internal masses in the loop, e.g.



New Frontiers

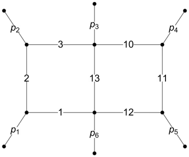
- 3-loop: $pp \to V + j$ [leading color], dijet, diphoton.





[Caola et al '21; Gehrmann et al '23]

- Some integrals for 4-jet production at NNLO.



[Henn et al '24]

Summary

- Amplitudes are a key bottleneck in making predictions.
- Modern understanding of perturbation theory has reached far.
- Tree-level/one-loop well understood (numerical algorithms).
- Frontiers:
  - Two-loop: new theoretical challenges. Results almost always analytic. Active area of research.
  - $\geq$ Three-loop in infancy. (No N^3LO $2 \to 2$ yet).