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Gravitational Waves Project

Consider a system of two compact stars (black holes or neutron stars) with masses m_1 and m_2 in a circular orbit. Such a system will lose energy by gravitational-wave (GW) emission and the stars will spiral inwards such that the orbital frequency increases with time following Kepler's third law. The emitted gravitational waveform is a "chirp" (similar to the chirping of birds) with both amplitude and frequency increasing with time. When the stars are widely separated, the problem can be treated perturbatively. In the leading order post-Newtonian approximation, the observed GW signal, which is a linear combination of the two polarizations $h_+(t)$ and $h_\times(t)$, can be computed as:

$$h(t) = A(t)\cos\varphi(t). \tag{1}$$

The amplitude $A(t)$ depends on a particular combination of the masses, called the *chirp mass* $\mathcal{M}_c$, the instantaneous frequency $F(t)$ of GWs, the luminosity distance d_L to the source, and a geometric factor $\mathcal{C}$ that depends on the location of the source in the sky and its orientation with respect to the detector.

$$A(t) = \mathcal{C}\ \frac{4\mathcal{M}_c^{5/3}\pi^{2/3}F(t)^{2/3}}{d_L}. \tag{2}$$

For simplicity, we shall assume $\mathcal{C} = 1$ which implies that the binary is conveniently oriented giving circular polarization and the source is located along the direction where the detector shows maximum directional sensitivity. The chirp mass can be expressed in terms of the total mass $M \equiv m_1 + m_2$ and reduced mass $\mu \equiv m_1 m_2/M$ as $\mathcal{M}_c = \mu^{3/5}M^{2/5}$. The frequency evolution $F(t)$ is given by

$$F(t) = \frac{(\mathcal{M}_c F_0^9)^{1/8}}{\left[(\mathcal{M}_c F_0)^{1/3} - 256\,F_0^3\mathcal{M}_c^2\pi^{8/3}\,t/5\right]^{3/8}} \tag{3}$$

where F_0 is the starting frequency of the signal: $F_0 \equiv F(t = 0)$. It can be seen that the frequency sweeps from lower to higher frequencies, until the approximation breaks down at $t = t_c$. The *coalescence time* t_c can be computed as

$$t_c = \frac{5}{256\,(\pi F_0)^{8/3}\mathcal{M}_c^{5/3}}. \tag{4}$$

Finally, the phase $\varphi(t)$ of the GW signal can be expressed as

$$\varphi(t) = \varphi_0 - 2\left(\frac{1}{256\,(\pi\mathcal{M}_c F_0)^{8/3}} - \frac{t}{5\mathcal{M}_c}\right)^{5/8}, \tag{5}$$

where φ_0 is the phase at $t = 0$. Waveforms computed in this lowest order approximation are sometime called a "Newtonian chirp".

Units: All expressions in this Section are written in geometrized units, in which $G = c = 1$. Mass and distance have units of seconds. Physical units can be obtained by replacing a mass $\mathcal{M}_c$ by $G\mathcal{M}_c/c^3$, and a distance d_L by cd_L. In geometrized units, $1M_\odot = 4.92549095 \times 10^{-6}$ s and $1\text{pc} = 1.0292712503 \times 10^8$ s. A sample Newtonian chirp waveform is shown in Fig. 1.

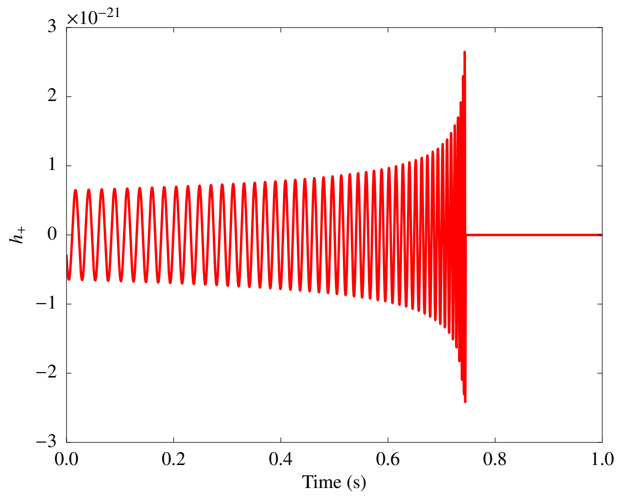

Figure 1: An example of a "Newtonian chirp", with chirp mass $\mathcal{M}_c = 10M_\odot$, distance $d_L = 100$ Mpc, initial phase $\varphi_0 = 0$ and start frequency $F_0 = 40$ Hz.

Matched filter

In the case a known signal $h(t)$ buried in stationary Gaussian, white noise, the optimal technique for signal extraction is the *matched filtering*, which involves cross-correlating the data with a *template* of the signal. The correlation function between two time series $x(t)$ and $\hat{h}(t)$ for a time shift τ is defined as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\,\hat{h}^\star(t)\,dt. \tag{6}$$

Above, * denotes complex conjugation, and $\hat{h(t)} \equiv h(t)/||h||$, where the norm $||h||$ of the template is defined by

$$||h||^2 = \int_0^{t_c} |h(t)|^2/\sigma^2\,dt,$$

where σ^2 is the variance of the noise. The optimal signal-to-noise ratio (SNR) is obtained when the template exactly matches with the signal.

$$\text{SNR} = ||h|| \tag{7}$$

If the SNR is greater than a predetermined threshold (which corresponds to an acceptably small false alarm probability), a detection can be claimed. Note that the actual detector data is neither white and is only approximately Gaussian, which makes actual GW detection a siginificantly more complex exercise than mentioned above!

Part 1. Write a code to generate the Newtonian chirp waveform $h(t)$ for arbitrary values of $\mathcal{M}_c, d_L, \varphi_0$.

Part 2. A data set `gw_ex_data_N.dat` containing a Newtonian GW signal with $d_L = 100$ Mpc, $\varphi_0 = 0$, $F_0 = 40\,\text{Hz}$, but unknown $\mathcal{M}_c$ is attached. The data $d(t)$ is comprised of the signal $h(t)$ and Gaussian white noise $n(t)$ of standard deviation $\sigma = 10^{-21}$. That is, $d(t) = h(t)+n(t)$. Write a code to detect the signal using the simple matched filtering method mentioned in the previous section. Since you don't know the chirp mass of the signal, choose a grid of chirp masses in the interval $\mathcal{M}_c \in (8, 12)M_\odot$ with some appropriate grid spacing. This is your "template bank"!

Working with detector data

For this part, it is recommended to use `gwpy`. You can download and install the package following the instructions at https://gwpy.github.io/docs/stable/. An excellent set of tutorials on data analysis with public LIGO-Virgo-KAGRA data is also available there.

Part 3. Download some data stretches for O1, O2, and O3 LIGO data. Compute the power spectral density of the detector noise, and check how the sensitivities of the instruments are improving with the observing runs. Note that the sensitivity is often quantified by a single sumber called the binary neutron star (BNS) range. Can you compute the typical BNS ranges for the O1, O2, and O3 LIGO instruments?

Part 4. Try to extend the matched filtering algorithm above for real detector data. What are the associated complications? Note that the algorithm is more appropriate for signals such as GW151226 than for GW150914. Why?